CHAPTER 4
NUMERICAL METHODS

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In this chapter some numerical techniques particularly useful in the field of machine design are briefly summarized. The presentations are directed toward automated calculation applications using electronic calculators and digital computers. The sequence of presentation is logically organized in accordance with the preceding table of contents, and emphasis is placed on useful equations and methods rather than on the derivation of theory.

4.1 NUMBERS

In the design and analysis of machines it is necessary to obtain quantities for various items of interest, such as dimensions, material properties, area, volume, weight, stress, and deflection. Quantities for such items are expressed by numbers accompanied by the units of measure for a meaningful perspective. Also, numbers always have an algebraic sign, which is assumed to be positive unless clearly designated as negative by a minus sign preceding the number. The various kinds of numbers are defined in Sec. 2-7, which see.

4.1.1 Real Numbers, Precision, and Rounding

Any numerical quantity is expressed by a *real number* which may be classified as an integer, a rational number, or an irrational number. For practical purposes of calculation or manufacturing, it is often necessary to approximate a real number by a specified number of digits. For some cases, significant numbers may be useful, and the following relates to the obtainable degree of precision.

Degree of Precision. In machine design, real numbers are expressed by significant digits as related to practical considerations of accuracy in manufacturing and operation. For example, a dimension of a part may be expressed by four significant digits as 3.876 in, indicating for this number that the dimension will be controlled in manufacturing by a tolerance expressed in thousandths of an inch. As another example, the weight density of steel may be used as 0.283 lbm/in^3, indicating a level of accuracy associated with control in the manufacturing of steel stock. Both these examples illustrate numbers as basic terms in a design specification.

However, it is often necessary to analyze a design for quantities of interest using equations of various types. Generally, we wish to evaluate a dependent variable by an equation expressed in terms of independent variables. The degree of precision obtained for the dependent variable depends on the accuracy of the predominant term in the particular equation, as related to algebraic operations. In what follows, we will assume that the accuracy of the computational device is better than the number of significant figures in a determined value.

For addition and subtraction, the predominant term is the one with the least number of significant decimals. For example, suppose a dimension D in a part is determined by three machined dimensions A, B, and C using the equation $D = A + B - C$. Specifically, if the accuracy of each dimension is indicated by the significant digits in $A = 12.50$ in, $B = 1.062$ in, and $C = 12.375$ in, the predominant term is A, since it has the least number of significant decimals with only two. Thus D would be accurate to only two decimals, and we would calculate $D = A + B - C = 12.50 + 1.062 - 12.375 = 1.187$ in. We should then round this value to two decimals, giving $D = 1.19$ in as the determined value. Also, we note that D is accurate to only three significant figures, although A and B were accurate to four and C was accurate to five.

For multiplication and division, the predominant term is simply defined as the one with the least number of significant digits. For example, suppose tensile stress σ is to be calculated in a rectangular tensile bar of cross section b by h using the equation $\sigma = P/(bh)$. Specifically, if $P = 15\,000$ lb, and as controlled by manufacturing accuracy $b = 0.375$ in and $h = 1.438$ in, the predominant term is b, since it has only three significant digits. Incidentally, we have also assumed that P is accurate to at least three significant digits. Thus we would calculate $\sigma = P/(bh) = 15\,000/[0.375(1.438)] = 27\,816$ psi. We should then round this value to three significant digits, giving $\sigma = 27\,800$ psi as the determined value.

For a more rigorous approach to accuracy of dependent variables as related to error in independent variables, the theory of relative change may be applied, as explained in Sec. 4.4.

Rounding. In the preceding examples, we note that determined values are rounded to a certain number of significant decimals or digits. For any case, the calculations are initially made to a higher level of accuracy, but rounding is made to give a more meaningful answer. Hence we will briefly summarize the rules for rounding as follows:

1. If the least significant digit is immediately followed by any digit between 5 and 9, the least significant digit is increased in magnitude by 1. (An exception to this rule is the case where the least significant digit is even and it is immediately fol-

lowed by the digit 5 with all trailing zeros. In that event, the least significant digit is left unchanged.)

2. If the least significant digit is immediately followed by any digit between 0 and 4, the least significant digit is left unchanged.

For example, with three significant digits desired, 2.765 01 becomes 2.77, 2.765 becomes 2.76, −1.8743 becomes −1.87, −0.4926 becomes −0.493, and 0.003 792 8 becomes 0.003 79.

4.1.2 Complex Numbers

Complex numbers are ones that contain two independent parts, which may be represented graphically along two independent coordinate axes. The independent components are separated by introduction of the operator $j = \sqrt{-1}$. Thus we express complex number $c = a + bj$, where a and b by themselves are either integers, rational numbers, or irrational numbers. Often a is called the *real component* and bj is called the *imaginary component*. The magnitude for c is $\sqrt{a^2 + b^2}$. For example, if $c = 3.152 + 2.683j$, its magnitude is

$$|c| = \sqrt{(3.152)^2 + (2.683)^2} = 4.139$$

Algebraically, the values for a and b may be positive or negative, but the magnitude of c is always positive.

4.2 FUNCTIONS

Functions are mathematical means for expressing a definite relationship between variables. In numerical applications, generally the value of a dependent variable is determined for a set of values of the independent variables using an appropriate functional expression. Functions may be expressed in various ways, by means of tables, curves, and equations.

4.2.1 Tables

Tables are particularly useful for expressing discrete value relations in machine design. For example, a catalog may use a table to summarize the dimensions, weight, basic dynamic capacity, and limiting speed for a series of standard roller bearings. In such a case, the dimensions would be the independent variables, whereas the weight, basic dynamic capacity, and limiting speed would be the dependent variables.

For many applications of machine design, a table as it stands is sufficient for giving the numerical information needed. However, for many other applications requiring automated calculations, it may be appropriate to transform at least some of the tabular data into equations by curve-fitting techniques. For example, from the tabular data of a roller-bearing series, equations could be derived for weight, basic dynamic capacity, and limiting speed as functions of bearing dimensions. The equations would then be used as part of a total equation system in an automated design procedure.

4.2.2 Curves

Curves are particularly useful in machine design for graphically expressing continuous relations between variables over a certain range of practical interest. For the case of more than one independent variable, families of curves may be presented on a single graph. In many cases, the graph may be simplified by the use of dimensionless ratios for the independent variables. In general, curves present a valuable picture of how a dependent variable changes as a function of the independent variables.

For example, for a stepped shaft in pure torsion, the stress concentration factor K_{ts} is generally presented as a family of curves, showing how it varies with respect to the independent dimensionless variables r/d and D/d. For the stepped shaft, r is the fillet radius, d is the smaller diameter, and D is the larger diameter.

For many applications of machine design, a graph as it stands may be sufficient for giving the numerical data needed. However, for many other applications requiring automated calculations, equations valid over the range of interest may be necessary. The given graph would then be transformed to an equation by curve-fitting techniques. For example, for the stepped shaft previously mentioned, stress concentration factor K_{ts} would be expressed by an equation as a function of r, d, and D derived from the curves of the given graph. The equation would then be used as part of a total equation system in the decision-making process of an automated design procedure.

4.2.3 Equations

Equations are the most powerful means of function expression in machine design, especially when automated calculations are to be made in a decision-making procedure. Generally, equations express continuous relations between variables, where a dependent variable y is to be numerically determined from values of independent variables x_1, x_2, x_3, etc. Some commonly used types of equations in machine design are summarized next.

Linear Equations. The general form of a linear equation is expressed as follows:

$$y = b + c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{4.1}$$

Constant b and coefficient $c_1, c_2, \ldots, c_n$ may be either positive or negative real numbers, and in a special case, any one of these may be zero.

For the case of one independent variable x, the linear equation $y = b + cx$ is graphically a straight line. In the case of two independent variables x_1 and x_2, the linear equation $y = b + c_1x_1 + c_2x_2$ is a plane on a three-dimensional coordinate system having orthogonal axes x_1, x_2, and y.

Polynomial Equations. The general form of a polynomial equation in two variables is expressed as follows:

$$y = b + c_1x + c_2x^2 + \cdots + c_nx^n \tag{4.2}$$

Constant b and coefficients $c_1, c_2, \ldots, c_n$ may be either positive or negative real numbers, and in a special case, any one of these may be zero.

For the special case of $n = 1$, the equation $y = b + c_1x$ is linear in x. For the special case of $n = 2$, the equation $y = b + c_1x + c_2x^2$ is known as a *quadratic equation.* For the special case of $n = 3$, the equation $y = b + c_1x + c_2x^2 + c_3x^3$ is known as a *cubic equation.* In general, for $n > 3$, Eq. (4.2) is known as a *polynomial of degree n.*

Simple Exponential Equations. The general form for a type of simple exponential equation commonly used in machine design is expressed as follows:

$$y = bx_1^{c_1}x_2^{c_2}\cdots x_n^{c_n} \tag{4.3}$$

Coefficient b and exponents $c_1, c_2, \ldots, c_n$ may be either positive or negative real numbers. However, except for the special case of any c_i being an integer, the corresponding values of x_i must be positive.

For the special case of $n = 1$ with $c_1 = 1$, the equation $y = bx$ is a simple straight line. For $n = 1$ with $c_1 = 2$, the equation $y = bx^2$ is a simple parabola. For $n = 1$ with $c_1 = 3$, the equation $y = bx^3$ is a simple cubic equation.

As a specific example of the more general case expressed by Eq. (4.3), a simple exponential equation might be as follows:

$$y = 38.69\frac{x_1^{2.670}x_4^2}{x_2^{0.092}x_3^{1.07}}$$

For this example, $n = 4$, $b = 38.69$, $c_1 = 2.670$, $c_2 = -0.092$, $c_3 = -1.07$, and $c_4 = 2$. Also, if at a specific point we have $x_1 = 4.321$, $x_2 = 3.972$, $x_3 = 8.706$, and $x_4 = 0.0321$, the equation would give the value of $y = 0.1725$.

The general form for another type of simple exponential equation occasionally used in machine design is expressed as follows:

$$y = bc_1^{x_1}c_2^{x_2}\cdots c_n^{x_n} \tag{4.4}$$

Coefficient b and independent variables $x_1, x_2, \ldots, x_n$ may be either positive or negative real numbers. However, except for the special case of any x_i being an integer, the corresponding values of c_i must be positive.

Transcendental Equations. The most commonly encountered types of transcendental equations are classified as being either trigonometric or logarithmic. For either case, inverse operations may be desired. In general, *transcendental equations* determine a dependent variable y from the value of an independent variable x as the argument.

The basic *trigonometric equations* are $y = \sin x$, $y = \cos x$, and $y = \tan x$. The argument x may be any real number, but it should carry angular units of radians or degrees. For electronic calculators, the units for x are generally degrees. However, for microcomputers or larger electronic computers, the units for x are generally radians.

The basic *logarithmic equation* is $y = \log x$. However, in numerical applications, care must be exercised in recognizing the base for the logarithmic system used. For natural logarithms, the Napierian base $e = 2.718\ 281\ 8\ldots$ is used, and the inverse operation would be $x = e^y$. For common logarithms, the base 10 is used, and the inverse operation would be $x = 10^y$.

A special relationship of importance is recognized by taking the logarithm of both sides in the simple exponential Eq. (4.3), resulting in the following equation:

$$\log y = \log b + c_1 \log x_1 + c_2 \log x_2 + \cdots + c_n \log x_n \tag{4.5}$$

We see that this equation is analogous to linear Eq. (4.1) by replacing $y, b, x_1, x_2, \ldots, x_n$ of Eq. (4.1) with $\log y$, $\log b$, $\log x_1$, $\log x_2, \ldots, \log x_n$, respectively. Thus the equation $y = bx^c$ will plot as a straight line on log-log graph paper, regardless of the values for constants b and c.

Combined Equations. Some basic types of equations have now been summarized, and they will be applied later in techniques of curve fitting. However, any of the more complicated equations found in machine design may be considered as special combinations of the basic equations, with the terms related by algebraic operations. Such equations might be placed in the general classification of combined equations. As a specific example of a combined equation, a *polynomial equation* is merely the sum of positive simple exponential terms, each of which has the general form of the right side of Eq. (4.3).

4.3 SERIES

A *series* is an ordered set of sequential terms generally connected by the algebraic operations of addition and subtraction. The number of terms can be either finite or infinite in scope. If the terms contain independent variables, the series is really an equation for calculating a dependent variable, such as the polynomial Eq. (4.2).

If a series is lengthy, it is often possible to approximate the series with a finite number of terms. The criterion for determining how many terms of the sequence are necessary is based on a consideration of convergence. The number of terms used must be sufficient for convergence of the determined value to an acceptable level of accuracy when compared with the entire series evaluation. This will be considered specifically in Sec. 4.4 on approximations and error.

Some commonly used series in machine design will be briefly summarized next. A more complete coverage can be found in any handbook on mathematics, and what follows is just a small sample.

4.3.1 Binomial Series

Consider the combined equation $y = (x_1 + x_2)^n$, where x_1 and x_2 are independent variables and n is an integer. The binomial series expansion of this equation is as follows:

$$\begin{aligned} y &= (x_1 + x_2)^n \\ &= x_1^n + nx_1^{n-1}x_2 + \frac{n(n-1)}{2!}x_1^{n-2}x_2^2 + \frac{n(n-1)(n-2)}{3!}x_1^{n-3}x_2^3 + \cdots \end{aligned} \tag{4.6}$$

In Eq. (4.6), if integer n is positive, the series consists of $n + 1$ terms. However, if integer n is negative, in general the number of terms is infinite and the series converges if $x_2^2 < x_1^2$.

4.3.2 Trigonometric Series

Some trigonometric relations will be approximated in Sec. 4.4 based on the series expansions summarized as follows:

$$y = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4.7}$$

$$y = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{4.8}$$

In Eqs. (4.7) and (4.8), angle x must be expressed in radians.

4.3.3 Taylor's Series

If any function $y = f(x)$ is differentiable, it may be expressed by a Taylor's series expansion as follows:

$$y = f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots \tag{4.9}$$

In Eq. (4.9), a is any feasible real number value of x, $f'(a)$ is the value of dy/dx at $x = a$, $f''(a)$ is the value of d^2y/dx^2 at $x = a$, and $f'''(a)$ is the value of d^3y/dx^3 at $x = a$. If only the first two terms in the series of Eq. (4.9) are used, we have a first-order Taylor's series expansion of $f(x)$ about a. If only the first three terms in the series of Eq. (4.9) are used, we have a second-order Taylor's series expansion of $f(x)$ about a. If $a = 0$ in Eq. (4.9), we have the special case known as a *Maclaurin's series expansion of* $f(x)$.

4.3.4 Fourier Series

Any periodic function $y = f(x) = f(x + 2\pi)$ can generally be expressed as a Fourier series expansion as follows:

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \tag{4.10}$$

where
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad \text{for } n = 0, 1, 2, 3, \ldots \tag{4.11}$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad \text{for } n = 1, 2, 3, \ldots \tag{4.12}$$

Coefficients a_n and b_n of Eq. (4.10) are determined by Eqs. (4.11) and (4.12).

For the Fourier series expansion of Eq. (4.10) to be valid, the Dirichlet conditions summarized as follows must be satisfied:

1. $f(x)$ must be periodic; i.e., $f(x) = f(x + 2\pi)$, or $f(x - \pi) = f(x + \pi)$.
2. $f(x)$ must have a single, finite value for any x.
3. $f(x)$ can have only a finite number of finite discontinuities and points of maxima and minima in the interval of one period of oscillation.

Techniques of numerical integration covered later can be applied to determine the significant Fourier coefficients a_n and b_n by Eqs. (4.11) and (4.12), respectively. A corresponding finite number of terms would then be used from the Fourier series of Eq. (4.10) for approximating $y = f(x)$. Fourier series are particularly valuable when complex periodic functions expressed graphically are to be approximated by an equation for automated calculation use.

4.4 APPROXIMATIONS AND ERROR

In many applications of machine design and analysis, it is advantageous to simplify equations by using approximations of various types. Such approximations are often obtained by using only the significant terms of a series expansion for the function.

The approximation used must give an acceptable degree of accuracy for the dependent variable over the range of interest for the independent variables. After defining error next, we will summarize some approximations particularly useful in machine design. Some other techniques of approximation will be presented later, under curve fitting, interpolation, root finding, differentiation, and integration.

4.4.1 Error

Relative error is defined as the difference between an approximate value and the true value, divided by the true value of a variable, as in Eq. (4.13):

$$e = \frac{y_a - y_t}{y_t} \tag{4.13}$$

From this equation, error e is determined as a dimensionless decimal, y_a is an approximate value for y, and y_t is the true value for y. If y_a and y_t are expressed by equations as functions of an independent variable x, Eq. (4.13) gives an error equation as a function of x.

Also, from Eq. (4.13) we see that error e carries an algebraic sign. For positive y_t, a positive value for e means that algebraically we have the relation $y_a > y_t$, whereas for negative e we would have $y_a < y_t$. The opposite relations are true if y_t is negative. Finally, the magnitude of error is its absolute value $|e|$.

For example, for $y_a = 1.003$ in and $y_t = 1.015$ in, by Eq. (4.13) we calculate $e = (1.003 - 1.015)/1.015 = -0.0118$. This means that y_a is 1.18 percent less than its true value y_t. The magnitude of the error is $|e| = 0.0118$.

Incidentally, if error occurs at random on two or more independent variables, the accompanying error on a dependent variable may be determined statistically. This will be illustrated specifically by application of the theory of variance, as presented later under relative change.

4.4.2 Arc Sag Approximation

Consider a circular arc of radius of curvature ρ as shown in Fig. 4.1 with sag y accompanying a chordal length of $2x$. The true value for y can be calculated from the following equation ([4.5], p. 60):

$$y_t = \rho\left[1 - \sqrt{1 - \left(\frac{x}{\rho}\right)^2}\right]$$

However, from the right triangle of Fig. 4.1, we obtain the following:

$$y_t = \frac{x^2 + y_t^2}{2\rho}$$

If in this equation we drop the term y_t^2, the following approximation is derived for y (its use would obviously simplify the calculation of either sag y or radius of curvature ρ):

$$y_a = \frac{x^2}{2\rho} \tag{4.14}$$

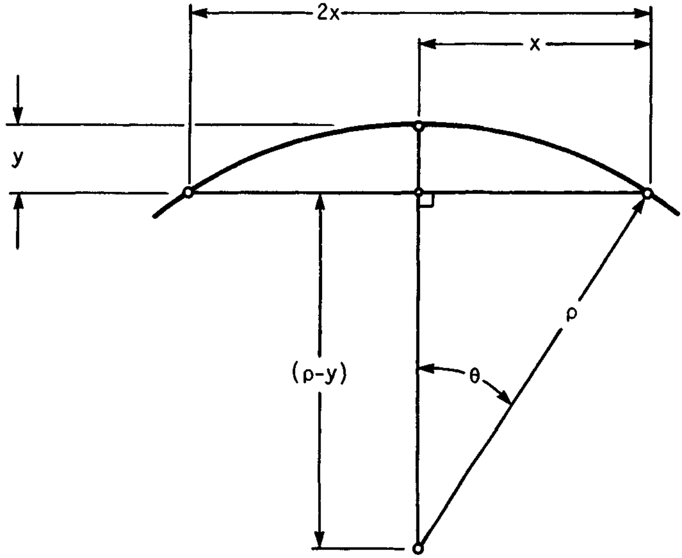

FIGURE 4.1 Circular arc of radius ρ showing sag y and chordal length $2x$.

Applying Eq. (4.13), error e in using approximate Eq. (4.14) is as follows ([4.5], p. 62):

$$e = \frac{-y_t}{2\rho} = -\sin^2 \frac{\theta}{2} \tag{4.15}$$

In Eq. (4.15), angle θ is as shown in Fig. 4.1. As specific examples, from this equation we find that y_a by Eq. (4.14) has error $e = -0.005$ for $\theta = 8.11°$, $e = -0.010$ for $\theta = 11.48°$, and $e = -0.02$ for $\theta = 16.26°$. Hence using the simple Eq. (4.14) to calculate sag would be acceptably accurate in many practical applications of machine design.

4.4.3 Approximation for 1/(1 ± *x*)

In some equations of analysis we have a term of the form $(1 + x)$ in the denominator. For purposes of simplification, as in operations of differentiation or integration, it may be desired not to have such a term in the denominator. Hence consider the true term $y_t = 1/(1 + x)$, which can be expanded into an infinite series by simple division, giving the following:

$$y_t = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

By dropping all but the first two terms of the series, $1/(1 + x)$ may be approximated by $1 - x$, expressed as follows:

$$\frac{1}{1+x} \approx y_a = 1 - x \tag{4.16}$$

Applying Eq. (4.13), the error in using this approximation is derived as follows:

$$e = \frac{y_a - y_t}{y_t}$$

$$= \frac{(1-x) - 1/(1+x)}{1/(1+x)}$$

$$e = -x^2 \tag{4.17}$$

As specific examples, for x within the range $-0.1 \le x \le 0.1$, we would have the corresponding error range of $-0.01 \le e \le 0$, whereas for $-0.02 \le x \le 0.2$ we would have $-0.04 \le e \le 0$.

Hence a denominator term of the form $1 + x$ could be replaced in an equation with a numerator term $1 - x$, providing the error is acceptably small over the anticipated range of variation for x. Similarly, a denominator term of the form $1 - x$ could be replaced with a numerator term $1 + x$ if the error is likewise acceptably small. The error equation in this case would still be Eq. (4.17).

4.4.4 Trigonometric Approximations

Approximations for some trigonometric functions will be summarized next, followed by the error function as derived by Eq. (4.13) in each case. For the summarized equations, angle x must be in radians. However, in the examples, ranges of angle x will be given in degrees, using the notation $x°$ in such cases.

An approximation for $\sin x$ is obtained by using only the first term in the Maclaurin's series of Eq. (4.7) as follows:

$$\sin x \approx x \tag{4.18}$$

$$e = \frac{x}{\sin x} - 1 \tag{4.19}$$

Hence for $-10° \le x° \le 10°$ we obtain positive error for e with $e \le 0.005\,10$, whereas for $-20° \le x° \le 20°$ we have positive error $e \le 0.0206$.

A more accurate approximation for $\sin x$ is obtained by using the first two terms in the series of Eq. (4.7) as follows:

$$\sin x \approx x - \frac{x^3}{6} \tag{4.20}$$

$$e = \frac{x}{\sin x}\left(1 - \frac{x^2}{6}\right) - 1 \tag{4.21}$$

Hence for $-50° \le x° \le 50°$ we obtain negative error for e with its magnitude $|e| \le 0.005\,41$.

An approximation for $\cos x$ is obtained by using only the first term in the Maclaurin's series of Eq. (4.8) as follows:

$$\cos x \approx 1 \tag{4.22}$$

$$e = \frac{1}{\cos x} - 1 \tag{4.23}$$

Hence for $-5° \leq x° \leq 5°$ we obtain positive error for e with $e \leq 0.003\,82$, whereas for $-15° \leq x° \leq 15°$ we have positive error $e \leq 0.0353$.

A more accurate approximation for $\cos x$ is obtained by using the first two terms in the series of Eq. (4.8) as follows:

$$\cos x \approx 1 - \frac{x^2}{2} \tag{4.24}$$

$$e = \frac{1 - x^2/2}{\cos x} - 1 \tag{4.25}$$

Hence for $-30° \leq x° \leq 30°$ we obtain negative error for e with its magnitude $e \leq 0.003\,58$.

An approximation for $\tan x$ is obtained by using only the first term of its Maclaurin's series expansion which follows:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Thus the approximation and error function are as follows:

$$\tan x \approx x \tag{4.26}$$

$$e = \frac{x}{\tan x} - 1 \tag{4.27}$$

Hence for $-10° \leq x° \leq 10°$ we obtain negative error for e with its magnitude $|e| \leq 0.0102$.

A more accurate approximation for $\tan x$ is obtained by using the first two terms in its series expansion as follows:

$$\tan x \approx x + \frac{x^3}{3} \tag{4.28}$$

$$e = \frac{x}{\tan x}\left(1 + \frac{x^2}{3}\right) - 1 \tag{4.29}$$

Hence for $-30° \leq x° \leq 30°$ we obtain negative error for e with its magnitude $|e| \leq 0.0103$.

4.4.5 Taylor's Series Approximations

Consider a general differentiable function $y = f(x)$. Its first-order Taylor's series approximation about $x = a$ is obtained by using only the first two terms of the Eq. (4.9) series, resulting in the following equation:

$$y = f(x) \approx f(a) + (x - a)f'(a) \tag{4.30}$$

In Eq. (4.30), a is any feasible real number value of x, and $f'(a)$ is the value of dy/dx at $x = a$.

The accuracy of Eq. (4.30) depends on the particular function $f(x)$ and the range anticipated for x about a. For this reason, a general error function is difficult to derive and impractical to apply. The clue for best accuracy is to choose a value for a such that $(x - a)$ will be small, resulting in negligible terms beyond the second in the Eq. (4.9) series.

For example, suppose we consider $f(x) = \sin x$ and anticipate a range of $-10° \leq x° \leq 10°$ for x. A good choice for a would be $a = 0$. Equation (4.30) would then give

$$\sin x \approx \sin 0 + x \cos 0 \qquad \therefore \sin x \approx x$$

This is merely Eq. (4.18), and the error analysis for the anticipated range of x has already been made after that equation.

However, if we still consider $f(x) = \sin x$ but anticipate a range of $45° \leq x° \leq 65°$ for x, Eq. (4.18) would be highly inaccurate. Hence Eq. (4.30) will be applied, and a good choice for a would be the midpoint of the x range, with $a = 55°(\pi/180) = 0.9599$ radian. Equation (4.30) would then give the following approximation:

$$\sin x \approx \sin 0.9599 + (x - 0.9599) \cos 0.9599 \qquad \therefore \sin x \approx 0.2685 + 0.5736x$$

Hence for $x° = 45°$ we would have $y_t = \sin 45° = 0.7071$ and $y_a = 0.2685 + 0.5736(45\pi/180) = 0.7190$. For that value of x, the error by Eq. (4.13) is

$$e = \frac{0.7190 - 0.7071}{0.7071} = 0.0168$$

For $x = 55°$ we would have $y_t = \sin 55° = 0.8192$ and $y_a = 0.2685 + 0.5736(55\pi/180) = 0.8191$. For that value of x, by Eq. (4.13), the error is

$$e = \frac{0.8191 - 0.8192}{0.8192} = -0.0001$$

Finally, for $x = 65°$ we would have $y_t = \sin 65° = 0.9063$ and $y_a = 0.2685 + 0.5736(65\pi/180) = 0.9192$. For that value of x, by Eq. (4.13), the error is

$$e = \frac{0.9192 - 0.9063}{0.9063} = 0.0142$$

For any differentiable $f(x)$, a more accurate approximation can be obtained by using the first three terms of the Eq. (4.9) series, giving a second-order Taylor's series approximation about $x = a$. The technique is similar to what has been illustrated for a first-order Taylor's series approximation. An appreciably greater range of accuracy would be achieved at the expense of increased complexity for the approximation derived.

4.4.6 Fourier Series Approximation

The Fourier series of Eq. (4.10) involves an infinite number of terms, and for practical calculations, only the significant ones should be used. The clue for significance is the relative magnitude of a Fourier coefficient a_n or b_n, since the amplitudes of $\sin nx$ and $\cos nx$ in Eq. (4.10) are both unity regardless of n.

In establishing significance of a Fourier coefficient, Eqs. (4.11) and (4.12) are solved, perhaps automatically by a computer using numerical integration. The Fourier coefficients are determined for $n = 1, 2, 3, \ldots, N$, where generally a value of N equal to 10 or 12 is sufficient for the investigation. Only the coefficients of significant relative magnitude for a_n and b_n are retained. They determine the significant harmonic content of the periodic function $f(x)$, and only those coefficients are used in the Eq. (4.10) series for the approximation derived. An error analysis could then be made for the derived approximation, including perhaps a graphic presentation by a computer video display for comparative purposes.

As a final item of practical importance, a Fourier series approximation can be derived for many nonperiodic functions $f(x)$ if independent variable x is limited to a definite range corresponding to 2π. In such a case, the derived approximation is used for calculation purposes only within the confined range for x. Hence the derivation assumes hypothetical periodicity outside the confined x range. Of course, the Dirichlet conditions previously stated must be satisfied for $f(x)$ within that range.

4.4.7 Relative Change and Error Analysis

Consider a general differentiable function expressed as follows and used specifically for calculating dependent variable y in terms of independent variables $x_1, x_2, \ldots, x_n$:

$$y = f(x_1, x_2, \ldots, x_n) \tag{4.31}$$

By the theory of differentiation, we can write the following equation in terms of partial derivatives and differentials for the variables:

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + \cdots + \frac{\partial y}{\partial x_n} dx_n \tag{4.32}$$

Small changes $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in $x_1, x_2, \ldots, x_n$ can be substituted respectively for the differentials $dx_1, dx_2, \ldots, dx_n$ of Eq. (4.32). Thus we obtain an approximation for estimating the corresponding change in y, designated as Δy in the following equation:

$$\Delta y \approx \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial y}{\partial x_n} \Delta x_n \tag{4.33}$$

This equation can be used to estimate the change in y corresponding to small changes or errors in $x_1, x_2, \ldots, x_n$.

As an example of application for Eq. (4.33), consider the simple exponential Eq. (4.3), since many equations in machine design are of this general form. Application of Eq. (4.33) to Eq. (4.3) results in the following simple approximation ([4.5], pp. 67–69):

$$\frac{\Delta y}{y} \approx c_1 \frac{\Delta x_1}{x_1} + c_2 \frac{\Delta x_2}{x_2} + \cdots + c_n \frac{\Delta x_n}{x_n} \tag{4.34}$$

In this equation $\Delta y/y$, $\Delta x_1/x_1$, $\Delta x_2/x_2, \ldots$, $\Delta x_n/x_n$ are dimensionless ratios corresponding to relative changes in the variables of Eq. (4.3).

As a specific example of application for Eq. (4.34), suppose we are given the following simple exponential equation:

$$y = \frac{5.32 x_1^{1.62} x_3^2}{x_2^{2.86}} \tag{4.35}$$

If at a point of interest we have the theoretical values $x_1 = 3.796$, $x_2 = 1.095$, and $x_3 = 2.543$, then Eq. (4.35) results in a theoretical value of $y = 230.35$. Suppose that errors exist on the theoretical values of $x_1, x_2, \ldots, x_n$, specifically given as $\Delta x_1 =$

0.005, $\Delta x_2 = 0.010$, and $\Delta x_3 = -0.020$. By Eq. (4.34) we calculate the corresponding relative change in y of Eq. (4.35) as follows:

$$\frac{\Delta y}{y} \approx 1.62\,\frac{0.005}{3.796} + -2.86\,\frac{0.010}{1.095} + 2\,\frac{-0.020}{2.543} = -0.0397$$

Thus the given errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ would result in a corresponding error of $\Delta y \approx -0.0397(230.35) = -9.14$ on the theoretical value of $y = 230.35$.

In the manner illustrated by the preceding example, by application of Eq. (4.34), accuracy estimates can quickly be made for simple exponential equations of the Eq. (4.3) form. The worst possible combination of errors for $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ can be used to estimate the corresponding error Δy on the theoretical value for y. However, for cases where random errors are anticipated on the independent variables, a statistical approach is more appropriate. This will be considered next.

A Statistical Approach to Error Analysis. Consider a general differentiable function of several variables typically expressed by Eq. (4.31). Suppose that relatively small errors are anticipated on the theoretical values of the independent variables $x_1, x_2, \ldots, x_n$, with a normal distribution of relatively small spread on any theoretical value for each variable considered as the mean. Designate the standard deviation of the normal distribution for each variable respectively by $\sigma_{x_1}, \sigma_{x_2}, \ldots, \sigma_{x_n}$. Then, for most cases, dependent variable y would approximately have a corresponding normal distribution with standard deviation σy on its theoretical value.

$$(\sigma_y)^2 \approx \left(\frac{\partial y}{\partial x_1}\right)^2 (\sigma_{x_1})^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 (\sigma_{x_2})^2 + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 (\sigma_{x_n})^2 \tag{4.36}$$

Suppose each of the independent variables $x_1, x_2, \ldots, x_n$ has a normal distribution typically shown in Fig. 4.2 with theoretical value corresponding to the mean value $\bar{x}_i$ for variable x_i. Let Δx_i represent a tolerance band, as shown in Fig. 4.2, corresponding to, say, three standard deviations. If the tolerance band Δx_i corresponds to three standard deviations, 99.73 percent of the total population for x_i values would be within the range $x_i - \Delta x_i \le x_i + \Delta x_i$, and we would use the following relation:

$$\Delta x_i = 3\sigma_{xi} \qquad \text{for } i = 1, 2, \ldots, n \tag{4.37}$$

Combining Eq. (4.37) with Eq. (4.36) by eliminating σ_{x_i} for $i = 1, 2, \ldots, n$, and using the corresponding relation $\Delta y = 3\sigma_y$, we obtain the following:

$$(\Delta y)^2 \approx \left(\frac{\partial y}{\partial x_1}\right)^2 (\Delta x_1)^2 + \left(\frac{\partial y}{\partial x_2}\right)^2 (\Delta x_2)^2 + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 (\Delta x_n)^2 \tag{4.38}$$

In this equation, all the tolerance bands $\Delta y, \Delta x_1, \Delta x_2, \ldots, \Delta x_n$ would correspond to three standard deviations and would encompass 99.73 percent of the total population for each variable.

As an example of application of Eq. (4.38), we will consider the general linear equation expressed by Eq. (4.1). Hence by calculus we obtain $\partial y/\partial x_1 = c_1$, $\partial y/\partial x_2 = c_2, \ldots, \partial y/\partial x_n = c_n$. Substituting these relations in Eq. (4.38), we obtain the following approximation for use in the case of linear Eq. (4.1):

$$(\Delta y)^2 \approx (c_1 \Delta x_1)^2 + (c_2 \Delta x_2)^2 + \cdots + (c_n \Delta x_n)^2 \tag{4.39}$$

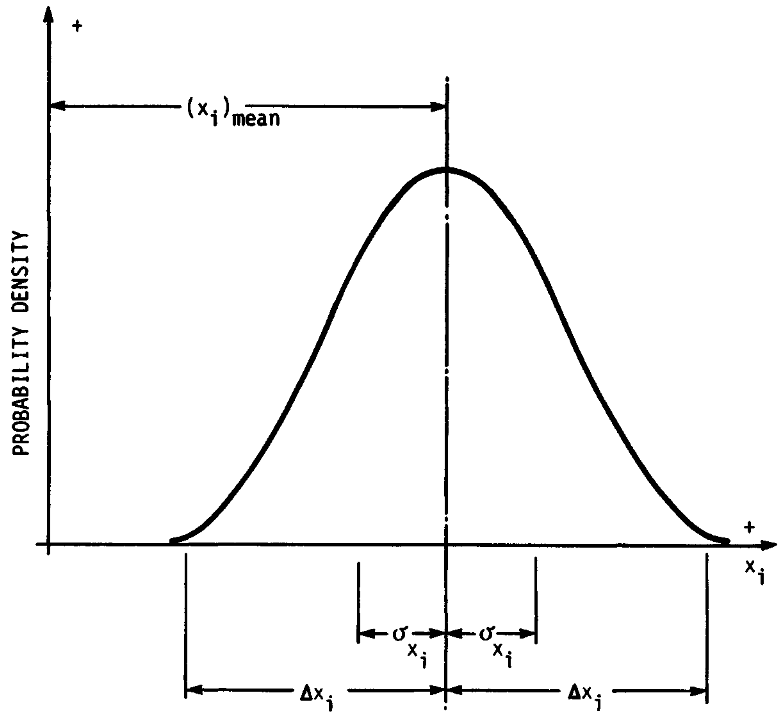

FIGURE 4.2 Typical normal distribution curve for an independent variable x_i.

As a specific example, suppose we have the following linear equation:

$$y = 2.97x_1 - 3.42x_2 + 7.81x_3$$

If tolerances of $\Delta x_1 = \pm 0.005$, $\Delta x_2 = \pm 0.015$, and $\Delta x_3 = \pm 0.010$ exist on the theoretical values of x_1, x_2, and x_3, we calculate the corresponding tolerance Δy on the theoretical value of y statistically by Eq. (4.39) as follows:

$$(\Delta y)^2 \approx [2.97(0.005)]^2 + [-3.42(0.015)]^2 + [7.81(0.010)]^2 \qquad \therefore\ \Delta y \approx \pm 0.0946$$

Thus the theoretical value of y calculated by the given linear equation would have a corresponding tolerance of $\Delta y \approx \pm 0.0946$. All the tolerances would correspond to, say, three standard deviations.

As another example of application for Eq. (4.38), we will consider the general simple exponential equation expressed by Eq. (4.3). By application of calculus to Eq. (4.3), we obtain the expressions for $\partial y/\partial x_1, \partial y/\partial x_2, \ldots, \partial y/\partial x_n$, which are then substituted into Eq. (4.38). Dividing the left and right sides of this equation, respectively, by the left and right sides of Eq. (4.3), we obtain the following approximation for use in the case of simple exponential Eq. (4.3):

$$\left(\frac{\Delta y}{y}\right)^2 \approx \left(\frac{c_1 \Delta x_1}{x_1}\right)^2 + \left(\frac{c_2 \Delta x_2}{x_2}\right)^2 + \cdots + \left(\frac{c_n \Delta x_n}{x_n}\right)^2 \tag{4.40}$$

As a specific example, suppose we are given the same simple exponential equation as before in Eq. (4.35). At a point of interest, we have the same theoretical values as before for $x_1, x_2, \ldots, x_n$ and y, and they are given following Eq. (4.35). However, the tolerance bands are now given as $\Delta x_1 = \pm 0.005$, $\Delta x_2 = \pm 0.010$, and $\Delta x_3 = \pm 0.020$, all corresponding to three standard deviations. Using the stated values following Eq. (4.40), we calculate statistically the corresponding tolerance Δy on the theoretical value of y as follows:

$$\left(\frac{\Delta y}{230.35}\right)^2 \approx \left[\frac{1.62(0.005)}{3.796}\right]^2 + \left[\frac{-2.86(0.010)}{1.095}\right]^2 + \left[\frac{2(0.020)}{2.543}\right]^2 \qquad \therefore\ \Delta y \approx \pm 7.04$$

Thus the theoretical value of y calculated by Eq. (4.35) as $y = 230.35$ would have a tolerance of $\Delta y \approx \pm 7.04$, corresponding to three standard deviations. As a final note, based on the given possibilities, we calculated $\Delta y \approx -9.47$ in the example following Eq. (4.35). However, based on probabilities, we have calculated $\Delta y \approx \pm 7.04$ in the present example.

4.5 FINITE-DIFFERENCE APPROXIMATIONS

Consider the general differentiable function $y = f(x)$ graphically shown in Fig. 4.3. First and second derivatives can be approximated at a point k of interest by the application of finite-difference equations. The simplest finite-difference approximations are summarized as follows ([4.5], pp. 28–35):

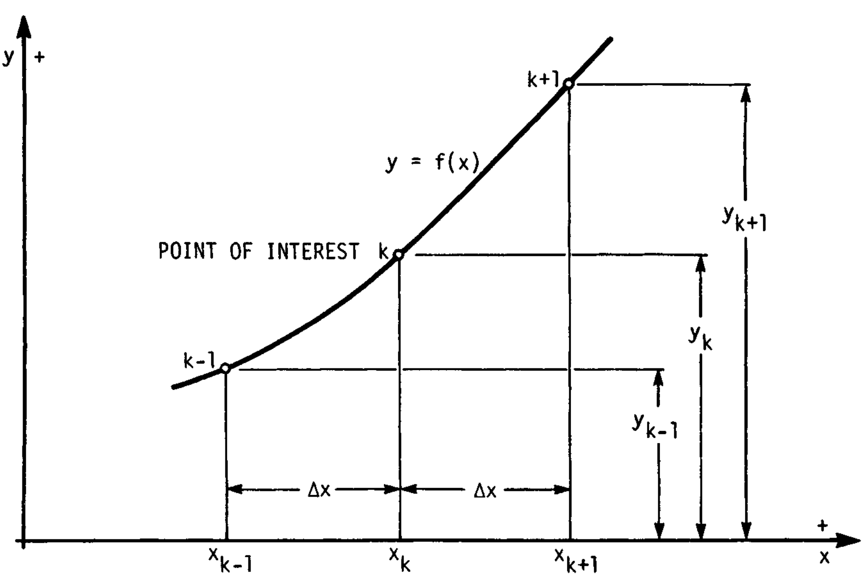

FIGURE 4.3 Graph of $y = f(x)$ showing three successive points used in finite difference equations.

$$\left(\frac{dy}{dx}\right)_k \approx \frac{y_{k+1} - y_{k-1}}{2\Delta x} \tag{4.41}$$

$$\left(\frac{d^2y}{dx^2}\right)_k \approx \frac{y_{k-1} + y_{k+1} - 2y_k}{(\Delta x)^2} \tag{4.42}$$

Equations (4.41) and (4.42) approximate the first and second derivatives of y with respect to x, respectively, at $x = x_k$. For both equations, the point of interest at x_k is surrounded by two equally spaced points, at x_{k-1} and x_{k+1}. The equal increment of spacing is Δx. The values of $y = f(x)$ at the three successive points x_{k-1}, x_k, and x_{k+1} are y_{k-1}, y_k, and y_{k+1}, respectively.

For most differentiable functions $y = f(x)$, the given finite-difference equations are reasonably accurate if the following two conditions are satisfied:

1. Spacing increment Δx, in general, should be reasonably small.
2. The values for y_{k-1}, y_k, and y_{k+1} must carry enough significant figures to give acceptable accuracy in the difference terms of Eqs. (4.41) and (4.42).

Adequate smallness of Δx can be determined by trial, that is, by successively decreasing Δx until no significant difference is determined in the calculated derivatives.

As a very simple test example, consider the function $y = \sin x$. Suppose we wish to calculate first and second derivatives at $x_k^\circ = 35°$ using Eqs. (4.41) and (4.42). We arbitrarily choose the increment $\Delta x^\circ = 2°$, giving $x_{k-1}^\circ = 33°$ and $x_{k+1}^\circ = 37°$. Thus $y_{k-1} = \sin 33° = 0.544\,639$, $y_k = \sin 35° = 0.573\,576$, and $y_{k+1} = \sin 37° = 0.601\,815$. However, for Eqs. (4.41) and (4.42), increment Δx must be expressed in radians, giving $\Delta x = 2(\pi/180) = 0.034\,906\,6$ radian. Hence by Eq. (4.41) we calculate

$$\left(\frac{dy}{dx}\right)_k \approx \frac{0.601\,815 - 0.544\,639}{2\,\Delta x}$$

$$= \frac{0.057\,176}{2(0.034\,906\,6)}$$

$$= 0.818\,99$$

Also, by Eq. (4.42), we calculate

$$\left(\frac{d^2y}{dx^2}\right)_k \approx \frac{0.544\,639 + 0.601\,815 - 2(0.573\,576)}{(\Delta x)^2}$$

$$= \frac{0.000\,698}{(0.034\,906\,6)^2}$$

$$= -0.573$$

To check the accuracy of the approximations, for $y = \sin x$ we know by calculus that $dy/dx = \cos x$ and $d^2y/dx^2 = -\sin x$. Therefore, the theoretically correct derivatives are calculated as $(dy/dx)_k = \cos x = \cos 35° = 0.819\,15$ and $(d^2y/dx^2)_k = -\sin x = -\sin 35° = -0.5736$. We see that the finite-difference approximations were reasonably accurate, which could be further improved by reducing Δx° to, say, 1°.

Finite-difference approximations can also be used for solving differential equations. Equations (4.41) and (4.42) can be used to substitute for derivatives in such differential equations, also substituting $x = x_k$ where encountered. The range of interest for x is divided into small increments Δx. At each net point so obtained, the finite-difference–transformed differential equation is evaluated to determine the discrepancies of satisfaction, known as *residuals.* An iterative procedure is logically developed for successively relaxing the residuals by changing x values at the net points until the differential equation is approximately satisfied at each net point. Thus the solution function $y = f(x)$ is approximated at each net point by such a numerical technique. The iterative procedure of relaxation is greatly facilitated by using a digital computer.

As a final item, finite-difference Eqs. (4.41) and (4.42) may be applied to calculate partial derivatives for the case of a differentiable function of several variables. Hence, for the equation $y = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$, the first and second partial derivatives may be approximated as follows:

$$\left(\frac{\partial y}{\partial x_i}\right)_k \approx \frac{(y_{k+1} - y_{k-1})_i}{2\Delta x_i} \tag{4.43}$$

$$\left(\frac{\partial^2 y}{\partial x_i^2}\right)_k \approx \frac{(y_{k-1} + y_{k+1} - 2y_k)_i}{(\Delta x_i)^2} \tag{4.44}$$

In these equations, the difference terms are subscripted by i, indicating that only x_i is incremented by Δx_i for calculating y_{k-1} and y_{k+1}, holding the other independent variables $x_1, x_2, \ldots, x_n$ constant at their k point values.

4.6 NUMERICAL INTEGRATION

Often it is necessary to evaluate a definite integral of the following form, where $y = f(x)$ is a general integrand function:

$$I = \int_{x_0}^{x_n} y\, dx \tag{4.45}$$

For the case where $y = f(x)$ is a complicated function, numerical integration will greatly facilitate obtaining the solution. If software is available for a particular computational device, the program should be directly applied. However, a commonly used numerical technique will be described next as the basis for writing a special program if necessary.

A simple and generally very accurate technique for numerical integration is based on Simpson's rule, referring to Fig. 4.4 for what follows. First, the limit range for x, between x_0 and x_n, is divided into n equal intervals by Eq. (4.46), where n must be an even number:

$$\Delta x = \frac{x_n - x_0}{n} \tag{4.46}$$

The values of y are then calculated at each of the net points so determined, giving y_0, $y_1, y_2, \ldots, y_{n-2}, y_{n-1}, y_n$. Simpson's rule, given by Eq. (4.47), is then used to approximate the definite integral I of Eq. (4.45):

$$I \approx \frac{\Delta x}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})] \tag{4.47}$$

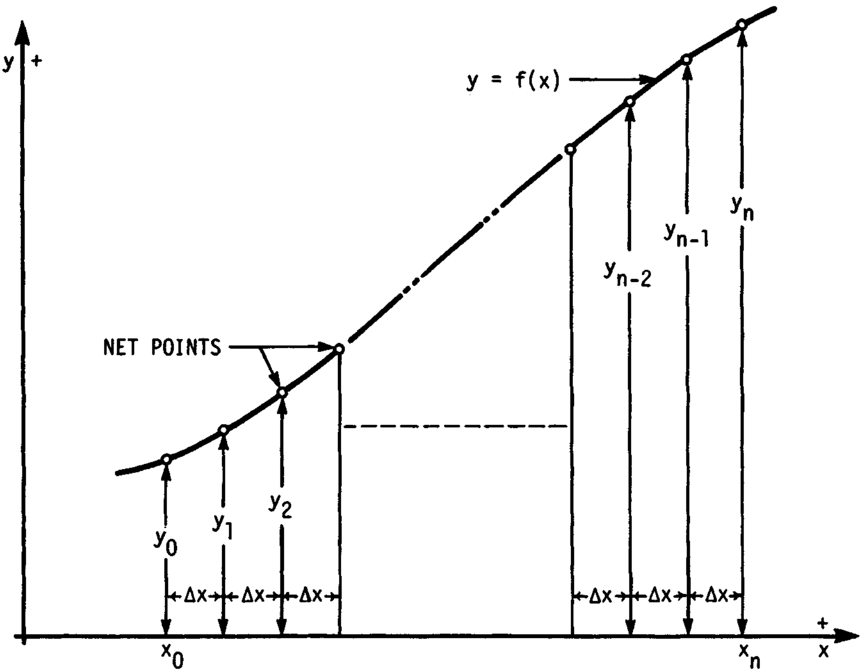

FIGURE 4.4 Graph of $y = f(x)$ divided into equal increments for numerical integration between x_0 and x_n by Simpson's rule.

With automated computation being used, probably the simplest way for determining adequacy of smallness for Δx is by trial. Hence even integer n is successively increased until the difference between successive I calculations is found to be negligible.

As a very simple test example, consider the following definite integral:

$$I = \int_{x_0}^{x_n} \sin x \, dx$$

Suppose the limits of integration are $x_0^\circ = 30°$ and $x_n^\circ = 60°$, giving $y_0 = \sin 30°$ and $y_n = \sin 60°$. For the test example, a value of $n = 20$ is arbitrarily chosen. Equation (4.46) is used to calculate Δx as follows, which must be expressed in radians for use in Eq. (4.47):

$$\Delta x = \frac{(60 - 30)(\pi/180)}{20} = 0.026\,179\,938\,8$$

In degrees, the increment is $\Delta x° = (60 - 30)/20 = 1.5°$. The y values at the remaining net points are then calculated as $y_1 = \sin 31.5°$, $y_2 = \sin 33°, \ldots, y_{n-2} = \sin 57°$, and $y_{n-1} = \sin 58.5°$. Simpson's rule is then applied using Eq. (4.47) to calculate the approximate value of $I = 0.366\,025\,404\,7$. The described procedure, of course, is programmed for automatic calculation, and specifically the TI-59 Master Library Program ML-09 was used for the test example [4.11]. To check the accuracy of the approximation, from elementary calculus we know that $\int \sin x\, dx$ is $-\cos x$. Hence, theoretically, we obtain $I = [(-\cos 60°) - (-\cos 30°)] = 0.366\,025\,403\,8$, and we see that the approximation for I by Simpson's rule was extremely accurate. See Sec. 5.4 for Richardson's error estimate when area is not known.

4.7 CURVE FITTING FOR PRECISION POINTS

Consider the situation where we have corresponding values of x and y available for a finite number of data points. Suppose we wish to derive an equation which passes precisely through some or all of these given data sets, and these we will call *precision points*. Some techniques of curve fitting for precision points will now be presented. In each case, accuracy checks could be made for the derived equation relative to all the given data points. Validity of the equation over the range of interest could then be established.

4.7.1 Simple Exponential Equation Curve Fit

In many cases of machine design, given graphs or tabular data would plot approximately as a straight line on log-log graph paper. Stress concentration factor graphs and a table of tensile strength versus wire diameter for spring steel are good examples. In such cases, a simple exponential equation of the following form can readily be derived for passing through two precision points (it is assumed that both x and y are positive):

$$y = bx^c \tag{4.48}$$

General curve shapes which are compatible with Eq. (4.48) are summarized graphically in Fig. 2.4 of Ref. [4.5]. Taking the logarithm of both sides in Eq. (4.48) results in the following, which reveals that a straight line would be the plot on log-log graph paper:

$$\log y = c \log x + \log b \tag{4.49}$$

Suppose two precision points (x_1, y_1) and (x_2, y_2) are chosen from the given data sets. The algebraic order for x is $x_1 < x_2$. If we use these precision points in Eq. (4.49), we obtain the following:

$$\log y_1 = c \log x_1 + \log b$$

$$\log y_2 = c \log x_2 + \log b$$

Subtracting the preceding two equations gives the following relation for calculating exponent c:

$$c = \frac{\log(y_2/y_1)}{\log(x_2/x_1)} \tag{4.50}$$

Either one of the two precision points can then be used to calculate coefficient b as follows, as derived from Eq. (4.48):

$$b = \frac{y_1}{x_1^c} = \frac{y_2}{x_2^c} \tag{4.51}$$

With values of c and b so determined, the simple exponential equation is uniquely defined.

As a simple example, suppose we have available the following data for two precision points:

x	y
0.1	8.5
0.25	5.3

Equation (4.50) would then yield the following value for exponent c:

$$c = \frac{\log(5.3/8.5)}{\log(0.25/0.1)} \qquad \therefore c = -0.516$$

Equation (4.51) would then give the following value for coefficient b:

$$b = \frac{8.5}{(0.1)^{-0.516}} = 2.591$$

Therefore, the derived equation passing through the given precision points is as follows:

$$y = \frac{2.591}{x^{0.516}}$$

Accuracy checks could then be made using Eq. (4.13) for all known data points to determine the validity of the derived equation over the range of interest.

4.7.2 Polynomial Equation Curve Fit

A polynomial equation of the following form can be derived to pass through $(n + 1)$ given precision points:

$$y = b + c_1 x + c_2 x^2 + \cdots + c_n x^n \tag{4.2}$$

The $(n + 1)$ given data sets are substituted into Eq. (4.2), giving $(n + 1)$ linear equations in terms of $b, c_1, c_2, \ldots, c_n$. These $(n + 1)$ linear equations are then solved simultaneously for the $(n + 1)$ unknowns $b, c_1, c_2, \ldots, c_n$, which uniquely defines the polynomial equation.

As a simple example, suppose we wish to derive a polynomial equation through the following four precision points. With $(n + 1) = 4$, we will obtain a polynomial equation of the third degree, since $n = 3$.

x	y
0.0	2.0
0.1	1.65
0.2	1.50
0.3	1.41

Substituting these data sets into Eq. (4.2), we obtain the following:

$$2.0 = b$$

$$\therefore 1.65 = 2.0 + c_1(0.1) + c_2(0.1)^2 + c_3(0.1)^3$$

$$1.50 = 2.0 + c_1(0.2) + c_2(0.2)^2 + c_3(0.2)^3$$

$$1.41 = 2.0 + c_1(0.3) + c_2(0.3)^2 + c_3(0.3)^3$$

Simultaneous solution of these linear equations gives $b = 2.0$, $c_1 = -4.97$, $c_2 = 17.0$, and $c_3 = -23.3$. Therefore, the derived polynomial equation passing through the four precision points is as follows:

$$y = 2.0 - 4.97x + 17.0x^2 - 23.3x^3$$

Accuracy checks could then be made using Eq. (4.13) for all known data points to determine the validity of the derived equation over the range of interest.

4.8 CURVE FITTING BY LEAST SQUARES

In many cases of machine design we wish to derive a simple equation $y = f(x)$ which approximates a large number of given data points (x_k, y_k) for $k = 1, 2, \ldots, M$, as illustrated in the following table:

x	y
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_k	y_k
$\vdots$	$\vdots$
x_M	y_M

The given data points are illustrated by + symbols in Fig. 4.5, which also shows the curve of the equation $y = f(x)$ to be derived. For any x_k, the difference between the given point value y_k and the corresponding equation value $f(x_k)$ is Δy_k, defined as follows:

$$\Delta y_k = y_k - f(x_k) \tag{4.52}$$

The equation $y = f(x)$ which minimizes the summation of $(\Delta y_k)^2$ terms for $k = 1$ to M of the given data set is known as the *least-squares fit.* A measure of accuracy for the derived equation is given by the dimensionless correlation coefficient r, which will have a value close to unity for the case of a "good" fit. Some simple examples will now be summarized for use in special cases of programming, although software programs are often already available for direct application [4.11].

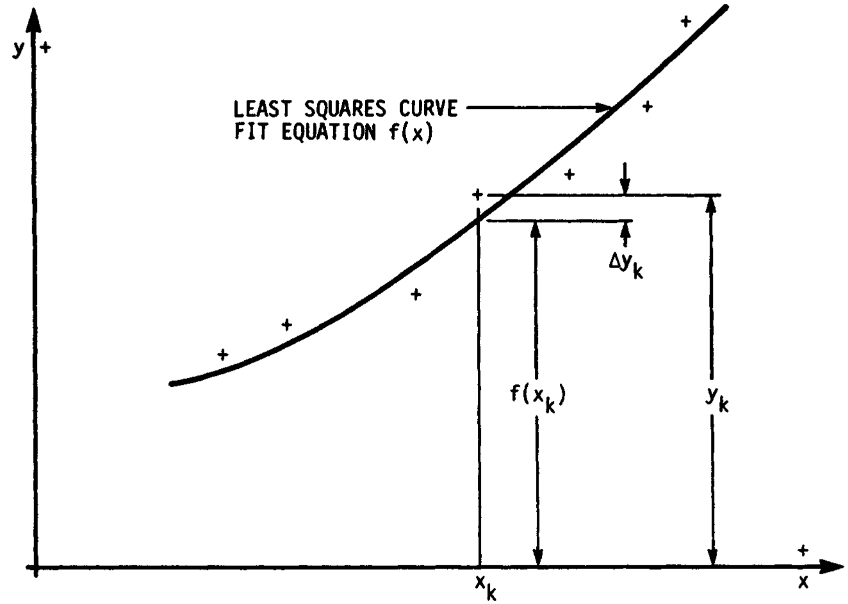

FIGURE 4.5 Least-squares curves $y = f(x)$ for given data points indicated by +.

4.8.1 Linear Equation Fit†

Consider the equation of a straight line as follows, which is to be used for curve fitting in the case where the given set of data points approximates a straight line on a graph:

$$y = b + cx \tag{4.53}$$

Such an equation can be made to pass through only two precision points. However, if many data points (x_k, y_k) are given, the least-squares fit is determined as follows: First, we calculate the values of five summations as follows for S_1 through S_5. In each case, the summations are made for $k = 1$ to M, corresponding to the given data points:

$$S_1 = \Sigma x_k \tag{4.54}$$

$$S_2 = \Sigma y_k \tag{4.55}$$

$$S_3 = \Sigma(x_k y_k) \tag{4.56}$$

$$S_4 = \Sigma(x_k^2) \tag{4.57}$$

$$S_5 = \Sigma(y_k^2) \tag{4.58}$$

† Ref. [4.5], pp. 55–56.

Then we calculate c and b for Eq. (4.53) by Eqs. (4.59) and (4.60), respectively:

$$c = \frac{MS_3 - S_1S_2}{MS_4 - S_1^2} \tag{4.59}$$

$$b = \frac{S_2 - cS_1}{M} \tag{4.60}$$

Finally, we calculate the correlation coefficient r as follows:

$$r = \frac{MS_3 - S_1S_2}{[(MS_4 - S_1^2)(MS_5 - S_2^2)]^{1/2}} \tag{4.61}$$

4.8.2 Simple Exponential Equation Fit†

Consider the simple exponential equation as follows, which is to be used for curve fitting in the case where the given set of data points approximates a straight line on a log-log graph:

$$y = bx^c \tag{4.48}$$

By taking the logarithm of both sides of this equation, we obtain the following:

$$\log y = \log b + c \log x \tag{4.49}$$

Hence, Eq. (4.48) would be a straight line on a log-log graph, and the least-squares fit is accomplished as follows: First, we calculate the values of three summations for S_1 through S_3 by Eqs. (4.62) to (4.64). In each case, the summations are made for $k =$ 1 to M, corresponding to the given data points:

$$S_1 = \Sigma(\log x_k) \tag{4.62}$$

$$S_2 = \Sigma(\log y_k) \tag{4.63}$$

$$S_3 = \Sigma[(\log x_k)(\log y_k)] \tag{4.64}$$

Then we calculate c and b for Eq. (4.48) by Eqs. (4.65) and (4.66), respectively:

$$c = \frac{MS_3 - S_1S_2}{2MS_1 - S_1^2} \tag{4.65}$$

$$\log b = \frac{S_2 - cS_1}{M} \tag{4.66}$$

Finally, we calculate the correlation coefficient r as follows:

$$r = \frac{MS_3 - S_1S_2}{[(2MS_1 - S_1^2)(2MS_2 - S_2^2)]^{1/2}} \tag{4.67}$$

As a specific example, suppose we are given the following set of data points:

† Ref. [4.5], pp. 56–57.

k	x_k	y_k
1	0.05	1.78
2	0.10	1.65
3	0.15	1.57
4	0.20	1.50
5	0.25	1.45
6	0.30	1.41

These data fall nearly as a straight line on a log-log graph, and Eq. (4.48) should be appropriate for a least-squares fit. Hence by Eqs. (4.62) to (4.67) we calculate the following values (we use $M = 6$, corresponding to the number of given data points):

$$c = -0.1305 \qquad b = 1.2138 \qquad r = 0.9929$$

Therefore, the derived equation for the least-squares fit is as follows:

$$y = \frac{1.2138}{x^{0.1305}}$$

We note that the correlation coefficient r is close to unity, so we conclude that the derived equation is a "good" fit.

4.8.3 Polynomial Equation Fit

Polynomial Eq. (4.1) may be used for a least-squares fit, but the derivation of such an equation is appreciably more complicated than the preceding examples. If interested, the designer should consult the literature for the details of derivation ([4.2], pp. 19–21).

4.9 CURVE FITTING FOR SEVERAL VARIABLES†

Occasionally in machine design we wish to derive a simple equation $y = f(x_1, x_2, \ldots, x_i, \ldots, x_n)$ for the case where we have n independent variables. In such cases, the problem of curve fitting can be very difficult. However, the following simple approach is often of acceptable accuracy in practical problems.

To start, consider the case of two independent variables x_1 and x_2, and we wish to derive an equation $y = f(x_1, x_2)$ to match approximately a given set of data points. Then the function $y = f(x_1, x_2)$ represents a three-dimensional surface using the orthogonal coordinate axes x_1, x_2, and y. The simple technique requires a common precision point for the given data, designated by subscript p in what follows. First, we derive an equation $y = f_1(x_1)$ by holding x_2 constant at $(x_2)p$. Next, we derive an equation $y = f_2(x_2)$ by holding x_1 constant at $(x_1)p$. The final equation is derived using $f_1(x_1)$ and $f_2(x_2)$ satisfying the y_p, $(x_1)_p$, and $(x_2)_p$ values of the given data.

As a simple specific example, consider the problem of deriving an equation $y = f(x_1, x_2)$ for given data-point values as follows:

† Ref. [4.5], pp. 57–59.

For $x_2 = 4.5$		For $x_1 = 3.0$	
x_1	y	x_2	y
2.0	3.0	2.5	5.4
3.0	4.2	4.5	4.2
5.0	6.4	7.0	3.5

The common precision points in these data are $(x_1)_p = 3.0$, $(x_2)_p = 4.5$, and $y_p = 4.2$. Plots of these data for y versus x_1 and y versus x_2 fall nearly as a straight line on log-log graphs. Hence Eq. (4.3) should be appropriate for the curve fit, giving the following form for the equation to be derived:

$$y = bx_1^{c_1}x_2^{c_2}$$

Thus the first and last data points are used for both parts of the table to calculate exponents c_1 and c_2 using Eq. (4.50) as follows:

$$c_1 = \frac{\log(6.4/3.0)}{\log(5.0/2.0)} = 0.8269$$

$$c_2 = \frac{\log(3.5/5.4)}{\log(7.0/2.5)} = -0.4212$$

Coefficient b for the equation is then calculated using the common precision-point values as follows:

$$4.2 = b(3.0)^{0.8269}(4.5)^{-0.4212} \qquad \therefore\ b = 3.190$$

Therefore, the derived equation for the curve fit is as follows:

$$y = 3.190\,\frac{x_1^{0.8269}}{x_2^{0.4212}}$$

Finally, accuracy checks could be made to see if the equation is acceptable for the intended use.

The simple technique as now illustrated can be applied to curve fitting for the case of more than two independent variables. Surprisingly, often a reasonably accurate equation is derived. As a specific example in machine design, see [4.5], pp. 383–388, for the derivation of an equation for helical gears having six independent variables. Accuracy checks are also presented in that example.

4.10 INTERPOLATION

Interpolation is generally of concern when we wish to estimate the value of a variable between two known data points. Suppose the values for x and y are known at two points k and $k + 1$ in Fig. 4.6. At some intermediate point j we wish to estimate either y_j for a specified value x_j or x_j for a specified value of y_j. The problem is really an application of curve fitting by using an equation $y = f(x)$ passing through the two precision points k and $k + 1$ for obtaining the estimate. Some specific techniques will be considered next.

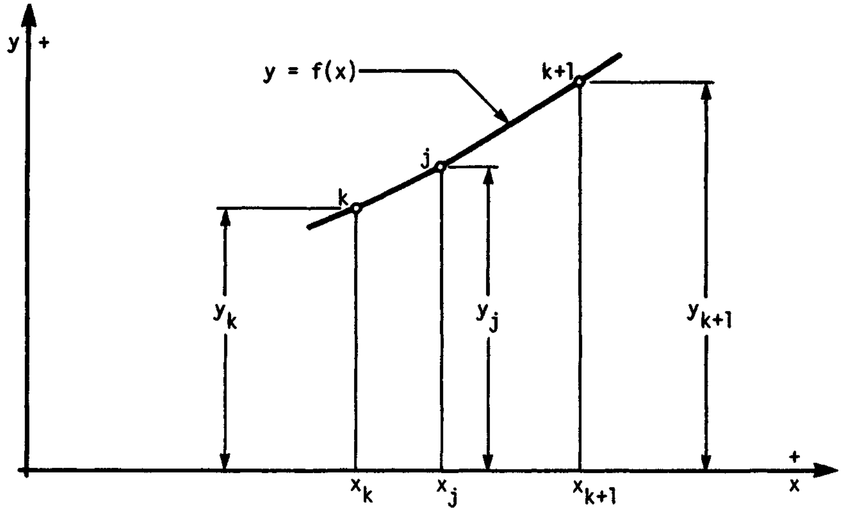

FIGURE 4.6 Curve $y = f(x)$ for interpolation at j between two data points k and $k + 1$.

4.10.1 Linear Interpolation

Consider passing a straight line between given points k and $k + 1$ in Fig. 4.6, with the assumed algebraic order for x being $x_k < x_j < x_{k+1}$. The equation for estimating y_j if x_j is specified would then be as follows:

$$y_j = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k}(x_j - x_k) \tag{4.68}$$

However, the equation for estimating x_j if y_j is specified would be as follows:

$$x_j = x_k + \frac{x_{k+1} - x_k}{y_{k+1} - y_k}(y_j - y_k) \tag{4.69}$$

In Eq. (4.69) it is assumed that y_k does not equal y_{k+1}, and y_j must be algebraically between y_k and y_{k+1}.

As a specific example, suppose we have the following values for x and y at two points:

Point	x	y
k	2.693	1.876
$k + 1$	2.981	2.210

For given $x_j = 2.729$, we would estimate y_j by Eq. (4.68) as follows:

$$y_j = 1.876 + \frac{2.210 - 1.876}{2.981 - 2.693}(2.729 - 2.693) \qquad \therefore\ y_j = 1.918$$

However, for given $y_j = 2.107$, we would estimate x_j by Eq. (4.69) as follows:

$$x_j = 2.693 + \frac{2.981 - 2.693}{2.210 - 1.876}(2.107 - 1.876) \qquad \therefore x_j = 2.892$$

4.10.2 Exponential Interpolation

If we believe that the given set of data points is curved and that it is compatible with a simple exponential type curve in the vicinity of precision points k and $k + 1$, we can apply Eqs. (4.50), (4.51), and (4.48) to derive Eqs. (4.70), (4.71), and (4.72), respectively, for use in the interpolation:

$$c = \frac{\log (y_{k+1}/y_k)}{\log (x_{k+1}/x_k)} \tag{4.70}$$

$$b = \frac{y_k}{x_k^c} \tag{4.71}$$

$$y_j = bx_j^c \tag{4.72}$$

It is assumed that x and y are both positive, with the algebraic order being $x_k < x_j < x_{k+1}$.

As a specific example we will use the preceding tabulated data for the points k and $k + 1$, and again we wish to estimate the value of y_j for given $x_j = 2.729$. Applying Eqs. (4.70), (4.71), and (4.72) we calculate the following values for c, b, and y_j, respectively:

$$c = \frac{\log (2.210/1.876)}{\log (2.981/2.693)} = 1.613$$

$$b = \frac{1.876}{(2.693)^{1.613}} = 0.3795$$

$$y_j = 0.3795(2.729)^{1.613} = 1.916$$

We see that this value of y_j is very close to the value of 1.918 previously calculated by linear interpolation.

4.11 ROOT FINDING

Given a function in the form $y = f(x)$, the problem is to find the values of x for which $y = 0$. For very simple functions, the roots can be found precisely. For example, for the given linear equation $y = b + cx$ we can choose any two values x_k and x_{k+1} and calculate the corresponding values y_k and y_{k+1}. Then, setting $y_j = 0$ in Eq. (4.69), we calculate the singular root x_j precisely. As another example, for the given parabolic equation $y = b + c_1x + c_2x^2$ we can find the two roots precisely using the quadratic equation as follows:

$$x = \frac{-c_1 \pm \sqrt{c_1^2 - 4bc_2}}{2c_2}$$

For more complicated functions $y = f(x)$, the problem of finding the roots becomes more difficult. Numerical methods may then be employed in an iterative procedure of automated calculation to approximate the values of x for which $y = 0$. For practical cases, to start, it is generally desired to determine the general characteristics of the complicated function $y = f(x)$, and this can be accomplished by execution of an exploratory search. From this initial stage, each root x_j will be bracketed in an interval $x_k \leq x_j \leq x_{k+1}$, with x and y values known at points k and $k + 1$. If only an approximate value for root x_j is needed, linear interpolation can be used by applying Eq. (4.69) directly and setting $y_j = 0$. However, if the root x_j is to be determined very accurately, an iterative numerical technique may be applied, such as interval halving or the Newton-Raphson method. The various procedures now mentioned will be outlined as follows.

4.11.1 Exploratory Search Stage

For complicated functions $y = f(x)$, it is advantageous to locate the approximate neighborhoods of the roots x_j before a more accurate determination is made for each. This is accomplished by an exploratory search stage, which calculates the y values at successive step points of the range of interest $x_{\min} \leq x \leq x_{\max}$. The exploratory search stage can be programmed in accordance with the flowchart in Fig. 4.7. Values of $x_{\min}$ and $x_{\max}$ are initially specified, as is the step increment Δx, which may require some trial. The increment Δx is chosen relatively large to save on computation time, but it must be small enough to identify the neighborhoods of the roots. These are recognized by an algebraic sign change in y for successive step points of the search. In this way, the roots are bracketed by known x and y values for step points which we will designate by subscripts k and $k + 1$ in what follows. First, we will consider a specific example for the exploratory search stage.

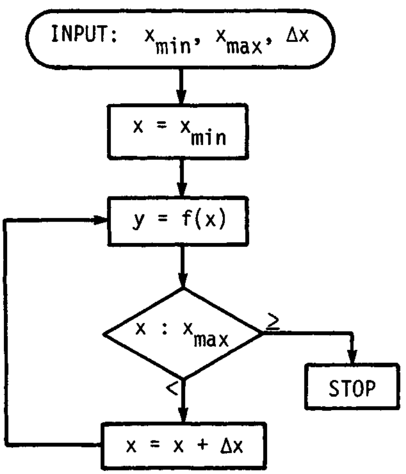

FIGURE 4.7 Exploratory search program for given equation $y = f(x)$.

Example. Consider the problem of finding the roots of the following equation:

$$y = \sqrt{x} - 3x^{1.53} + 7.656 \tag{4.73}$$

Suppose the range of interest is $0.2 \leq x \leq 3.8$, and we choose $\Delta x = 0.2$ for the exploratory search to be made in accordance with Fig. 4.7. The results are tabulated as follows:

x	y	x	y
0.2	7.848	2.0	0.407
0.4	7.550	2.2	–0.884
0.6	7.058	2.4	–2.246
0.8	6.418	2.6	–3.674
1.0	5.656	2.8	–5.168
1.2	4.786	3.0	–6.723
1.4	3.819	3.2	–8.338
1.6	2.763	3.4	–10.011
1.8	1.624	3.6	–11.741
		3.8	–13.525

Hence by the sign change in y we recognize that there is only one root to the given equation in the range of interest, and it will be located between $x_k = 2.0$ and $x_{k+1} = 2.2$. We now have the problem of determining more accurately the value of this root x_j.

4.11.2 Approximate Roots by Linear Interpolation

In many practical applications of machine design, it is only necessary to determine approximate values for the roots of an equation. In such cases, the roots are first bracketed by an exploratory search, and linear interpolation is then made between the bracketed points k and $k + 1$ for each root using Eq. (4.69) with $y_j = 0$. As a specific illustration, consider the preceding example where we found from the exploratory search that the root of Eq. (4.73) lies between $x_k = 2.0$ and $x_{k+1} = 2.2$. The corresponding values for y are calculated accurately by Eq. (4.73), giving $y_k =$ 0.406 638 and $y_{k+1} = -0.884\ 458$. These values are substituted in Eq. (4.69) with $y_j =$ 0, giving the approximate root as $x_j = 2.063$.

4.11.3 Roots by Interval Halving

From the exploratory search for $y = f(x)$, each root is bracketed within an original interval of uncertainty $[x_k, x_{k+1}]$, as shown in Fig. 4.8. The midpoint of this interval is then determined with respect to x, and y is calculated at that point by the given function $f(x)$. Thus a new interval of uncertainty is determined based on the sign of the calculated y, as shown in the figure. Its size is one-half the original interval. The process is successively repeated until the interval of uncertainty is reduced to a size Δx which is equal to or less than a specified accuracy ε on x. The described calculation strategy is summarized in the flowchart in Fig. 4.9. In general, as the search progresses, the values of x_j and y_j are known at point A in Fig. 4.8 for an interval of uncertainty, the midpoint value x_{j+1} is determined for point C in the figure, and y_{j+1} is calculated for that point by $f(x_{j+1})$. If the product $y_j y_{j+1}$ is positive, the new interval of uncertainty is as shown in Fig. 4.8. However, if the product $y_j y_{j+1}$ is negative, the new interval of uncertainty would be within the range $[x_j, x_{j+1}]$ of the figure. For each new interval of uncertainty, its span Δx is one-half of what it was previously, and only one function evaluation is necessary for its determination, in accordance with the flowchart in Fig. 4.9.

As a specific example, consider the problem of finding the root of Eq. (4.73) with the previously tabulated results from the exploratory search now available. There-

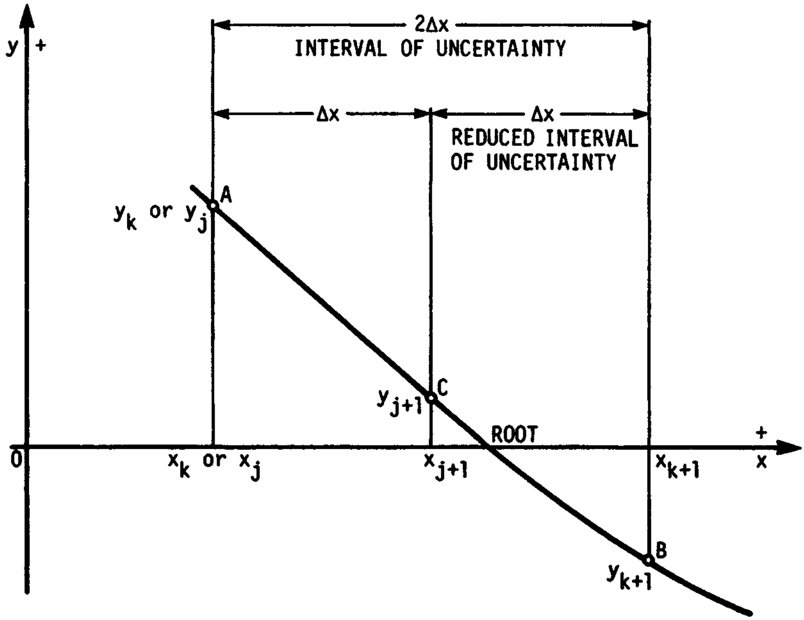

FIGURE 4.8 Reduction of interval of uncertainty by interval halving for root finding.

fore, for the input of Fig. 4.9, we would use $x_k = 2.0$ and $x_{k+1} = 2.2$, and we choose an accuracy specification of $\varepsilon = 10^{-6}$ for the root to be determined. The Fig. 4.9 calculation process was programmed on a TI-59 calculator, resulting in the following root value at the conclusion of the search (which took approximately 60 seconds for the execution time):

$$\text{Root} \approx x_{j+1} = 2.064\,209\,747$$

Incidentally, the corresponding value for y_{j+1} is $-7.125\,32 \times 10^{-7}$, which we see is very close to zero, as it should be for the root.

4.11.4 Roots by the Newton-Raphson Method

The Newton-Raphson method is generally a highly efficient iterative technique for very accurately finding the roots of a given complicated function $y = f(x)$. For the method to work with some given functions it is necessary to start the search process at a point not too far from the root. The exploratory search stage will give the function characteristics necessary for choosing a good starting point.

If in the iterative search process of the Newton-Raphson method we are at some point j in Fig. 4.10, we determine an improved estimate x_{j+1} for the root by extrapolation as follows:

$$\left(\frac{dy}{dx}\right)_j = f'(x_j) = \frac{y_j}{x_j - x_{j-1}} \qquad \therefore\ x_{j+1} = x_j - \frac{y_j}{f'(x_j)} \tag{4.74}$$

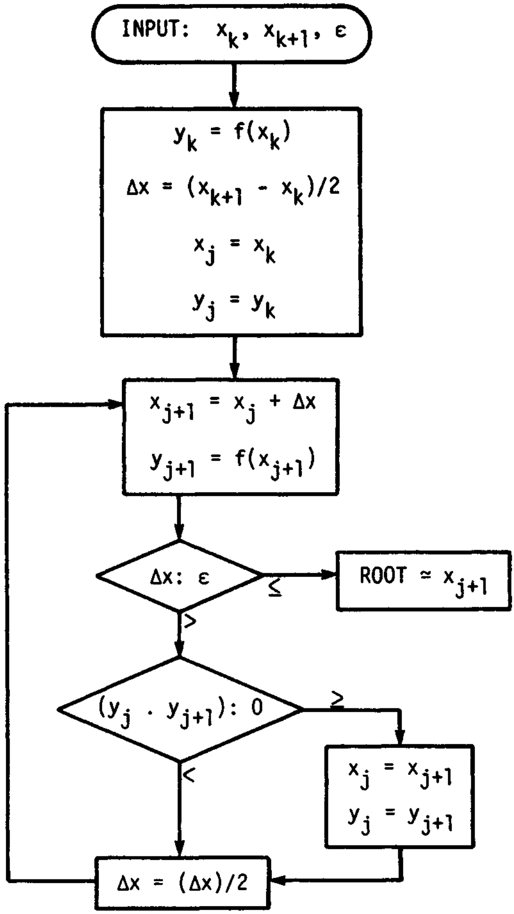

FIGURE 4.9 Interval-halving flowchart for finding root of $y = f(x)$ within bracketed interval $[x_k, x_{k+1}]$.

However, for complicated functions, the equation for $f'(x)$ is generally difficult to derive, and we circumvent this problem by resorting to a finite difference approximation instead. Thus consider an adjacent point A separated from j by a small increment δ, as shown in Fig. 4.10. Therefore, the finite difference approximation for $f'(x_j)$ is as follows:

$$f'(x_j) \approx \frac{y_A - y_j}{\delta}$$

where $y_A = f(x_j + \delta)$ and $y_j = f(x_j)$. Substituting this finite difference approximation into Eq. (4.74), we obtain Eq. (4.75) for estimating x_{j+1}:

$$x_{j+1} \approx x_j - \frac{\delta y_j}{y_A - y_j} \tag{4.75}$$

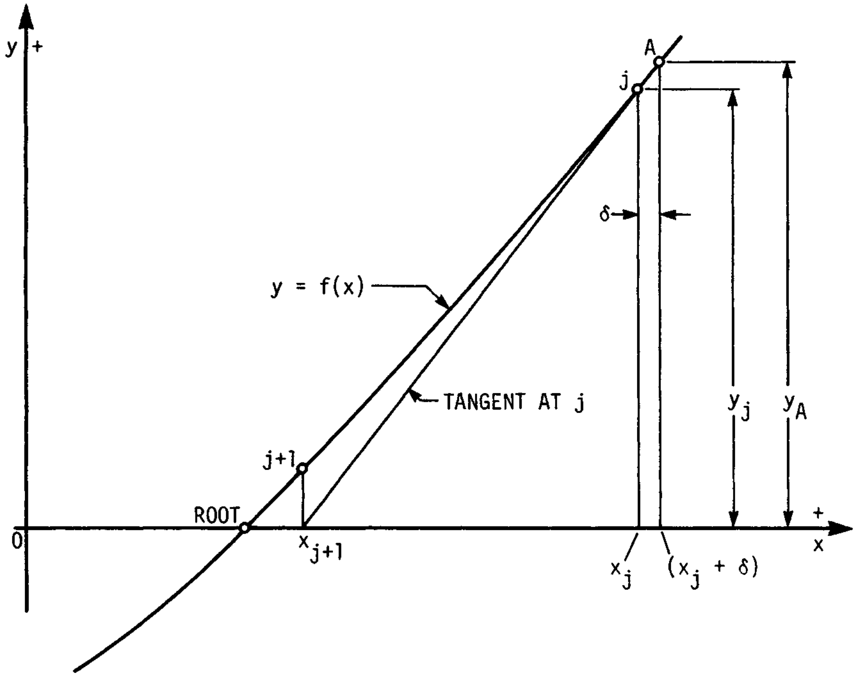

FIGURE 4.10 Newton-Raphson method of root finding.

For use of this approximation, two function evaluations are necessary for y_j and y_A as explained. The iterative calculation procedure for finding the root is summarized in the flowchart in Fig. 4.11, where start point x_k is initially specified. Also, solution accuracy ε and finite difference increment δ are generally specified as relatively small numbers compared with x_k.

As a specific example, consider the problem of finding the root of Eq. (4.73), with the previously tabulated results from the exploratory search now available. Therefore, for the input in Fig. 4.11 we could use $x_k = 2.0$ as a good start point. Also, we choose the accuracy specification for the root as $\varepsilon = 10^{-6}$ and finite difference increment as $\delta = 10^{-6}$. The Fig. 4.11 calculation process was programmed on a TI-59 calculator, resulting in the following root value at the conclusion of the search (which took approximately 15 seconds for the execution time):

$$\text{Root} \approx x_{j+1} = 2.064\ 209\ 636$$

Incidentally, the corresponding value for y_{j+1} is -1.054×10^{-9}, which we see is extremely close to zero, as it should be for the root. Finally, it should be mentioned that exactly the same root was found by the Fig. 4.11 program using other start points of $x_k = 0.2$, $x_k = 2.2$, and $x_k = 3.8$.

4.11.5 Summary of Roots Found for Eq. (4.73)

A comparison of the root findings for Eq. (4.73) from the preceding examples is given in Table 4.1. We see that linear interpolation was extremely fast, but the root

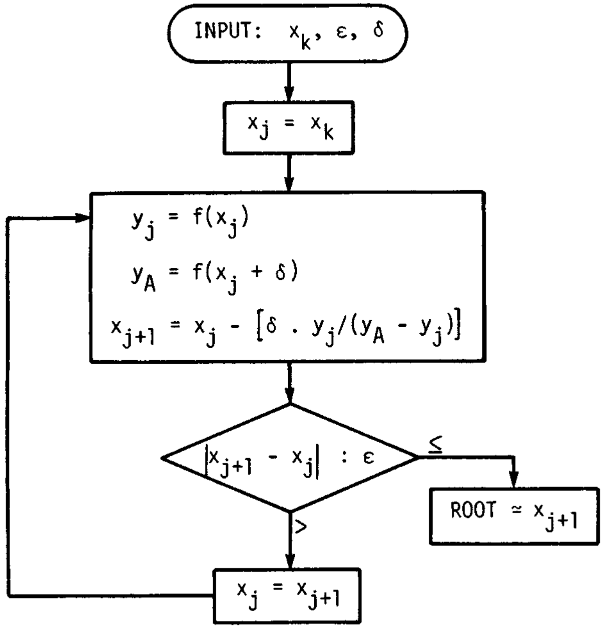

FIGURE 4.11 Flowchart for finding root of $y = f(x)$ by Newton-Raphson method.

was determined only approximately. The last two methods were extremely accurate, but more time-consuming for the solution. As indicated by the summary, the Newton-Raphson method is fast, but it does not provide range numbers as interval halving and golden section can. See Ref. [4.8], Chap. 8.

4.12 SYSTEMS OF EQUATIONS

The simultaneous solution of two or more equations can be a very difficult problem. In general, the number of unknowns cannot exceed the number of equations. We wish to find the common solution to the equation system, and some simple techniques will now be outlined.

TABLE 4.1 Root for Eq. (4.73)

Method	Root found	TI-59 execution time, s
Linear interpolation	2.063	2
Interval halving	2.064 209 747	60
Newton-Raphson	2.064 209 636	15

4.12.1 Two Equations with Two Unknowns

Case 1. Consider the problem where y is expressed by two given functions of x as follows:

$$y = f_1(x) \tag{4.76}$$

$$y = f_2(x) \tag{4.77}$$

We wish to find the common values of x and y at which the two curves cross in Fig. 4.12. To start, we could display the two curves graphically, to be sure that they do cross in the range of interest for x. Depending on the general characteristics of the functions, we could devise an iterative scheme for converging to the neighborhood of the solution point as schematically shown in Fig. 4.12. However, a more direct approach might be to equate the functions, defining $g(x)$ and giving the following single equation to work with:

$$g(x) = f_1(x) - f_2(x) = 0 \tag{4.78}$$

In this way, the problem has been simplified to one of merely finding the root of $g(x)$ expressed by Eq. (4.78).

As a specific example, suppose we are given the following two equations expressing y as a function of x:

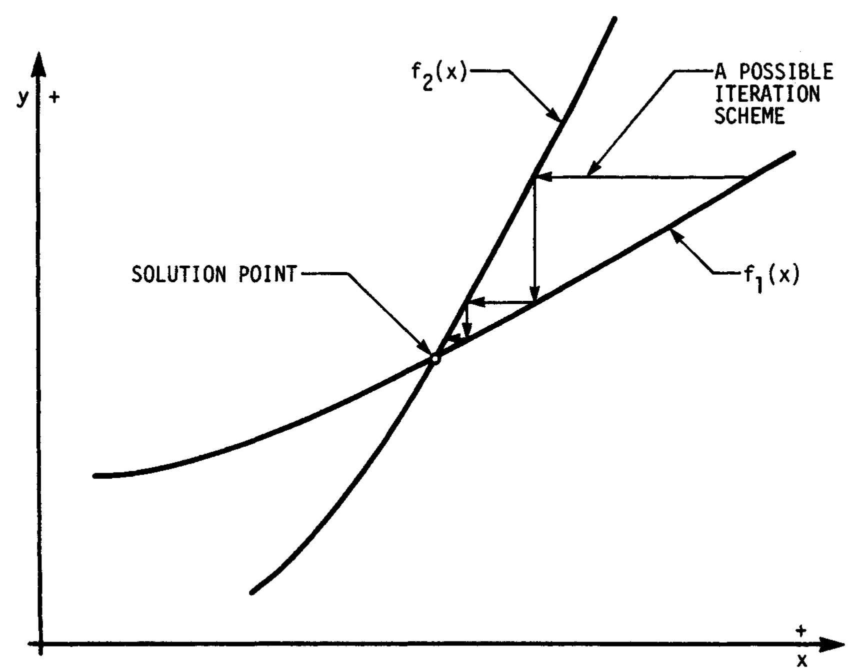

FIGURE 4.12 Simultaneous solution of two functions.

$$y = \sqrt{x} + 4.302 \tag{4.79}$$

$$y = 3x^{1.53} - 3.354 \tag{4.80}$$

By equating the two functions of x, we derive the following relation for obtaining the solution:

$$g(x) = \sqrt{x} - 3x^{1.53} + 7.656 = 0 \tag{4.81}$$

The solution value for x is merely the root of this equation, and a root-finding technique would be applied as previously explained. Since $g(x)$ of Eq. (4.81) is specifically the same as the right side of Eq. (4.73), from Table 4.1 the root is as follows:

$$x = 2.064\,210$$

The corresponding value for y can then be calculated by either Eq. (4.79) or Eq. (4.80), giving the following specifically by Eq. (4.79):

$$y = \sqrt{2.064\,210} + 4.302 = 5.739$$

The same value for y is obtained by Eq. (4.80).

Case 2. Consider the problem where we have two given functions of two variables x_1 and x_2 expressed in the following form:

$$f_1(x_1, x_2) = 0 \tag{4.82}$$

$$f_2(x_1, x_2) = 0 \tag{4.83}$$

As is often feasible, combine the two given equations by eliminating either x_1 or x_2. Thereby, a single equation in one variable is obtained, such as $g(x_1) = 0$ or $h(x_2) = 0$, which is solved by a root-finding technique as previously described. The corresponding remaining value for x_1 or x_2 is then readily calculated by reversing the equation-combination procedure.

As a specific example, suppose we are given the following two functions of x_1 and x_2:

$$\sqrt{x_1} + 3x_2 + 1.053 = 0 \tag{4.84}$$

$$x_1^{1.53} + x_2 - 2.201 = 0 \tag{4.85}$$

If we multiply Eq. (4.85) by 3 and subtract what is obtained from Eq. (4.84), we eliminate x_2 and obtain the following relationship for $g(x)$:

$$g(x) = \sqrt{x_1} - 3x_1^{1.53} + 7.656 = 0 \tag{4.86}$$

A root-finding technique would then be applied to Eq. (4.86), and since $g(x)$ is specifically the same as the right side of Eq. (4.73), we may use the results from Table 4.1 in this example. Thus we have found for the solution point the following value for x_1:

$$x_1 = 2.064\,210$$

The corresponding value for x_2 can then be calculated by either Eq. (4.84) or Eq. (4.85), giving the following specifically by Eq. (4.84):

$$x_2 = \frac{-(\sqrt{2.064\,210} + 1.053)}{3} = -0.829\,9$$

The same value for x_2 is obtained by Eq. (4.85).

For the situation where Eqs. (4.82) and (4.83) cannot readily be combined by eliminating either x_1 or x_2, curve-fitting techniques can generally be applied to either one of the equations, giving an explicit equation for either x_1 or x_2 expressed in terms of the other variable. This transformed equation is then substituted in the described equation-combination procedure for obtaining the solution.

4.12.2 Several Equations with Several Unknowns

The described procedures can generally be extended to solve several equations with several unknowns. For either the case 1 or case 2 type of problems previously described, the given equation system, now several in number, is reduced by equation combination to a single function of a single variable whose solution is found by a root-finding technique. The equation-combination procedure is then reversed to find the values of the other variables. If necessary, curve-fitting techniques may be employed to facilitate the equation-combination process.

4.13 OPTIMIZATION TECHNIQUES

In critical problems of design, the engineer wishes to make decisions which are as favorable as possible for the particular application at hand. In such cases, optimization of design is often worth striving for in the decision-making process. Based on the most critical aspects of the particular problem, an appropriate optimization objective must be chosen and mathematically formulated. Also, constraints of various types must be satisfied in almost all practical problems of design optimization, and these must be mathematically formulated. Hence the engineer must simultaneously address a complicated equation system of the following general form for arriving at decisions of optimal design:

$$Q^{\ddagger} = f(x_1, x_2, \ldots, x_i, \ldots, x_n) \tag{4.87}$$

subjected to

$$y_j = g_j(x_1, x_2, \ldots, x_i, \ldots, x_n) \qquad \text{for } j = 1, 2, \ldots, J \tag{4.88}$$

and

$$x_i \gtreqless c_i \qquad y_j \gtreqless c_j \tag{4.89}$$

In this compact representation of a complicated equation system, $Q^{\ddagger}$ of Eq. (4.87) is the optimization quantity to be either minimized or maximized, and $x_1, x_2, \ldots, x_i, \ldots, x_n$ are the independent variables. In Eq. (4.88), y_j represents a dependent variable, and there are J such equations in the system. Finally, the constraints of Eq. (4.89) are on the independent variables x_i and the dependent variables y_j. The general symbol $\gtreqless$ means that the required relation is one of the following, for any of the variables: $>$, $\geq$, $=$, $\leq$, or $<$. Specified constants c_i and c_j are the numerical limits imposed on the associated variables for an acceptable design, and any one may be zero in value.

Optimization of design is too large a subject area to describe in detail in this section. An explicit *method of optimal design* has been developed and applied to many practical examples of machine design, and it is particularly suited to mechanical elements and devices of various types (see Refs. [4.4] and [4.5]). By this technique, a calculation flowchart is derived for explicitly solving the optimization problem in numerical application. Many algorithms have also been developed based on techniques of nonlinear programming for iterative solutions to optimization problems by automated optimal design (see Refs. [4.3], [4.8], [4.9], and [4.12]).

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